Boundaries of Strata 101: WYSIWYG

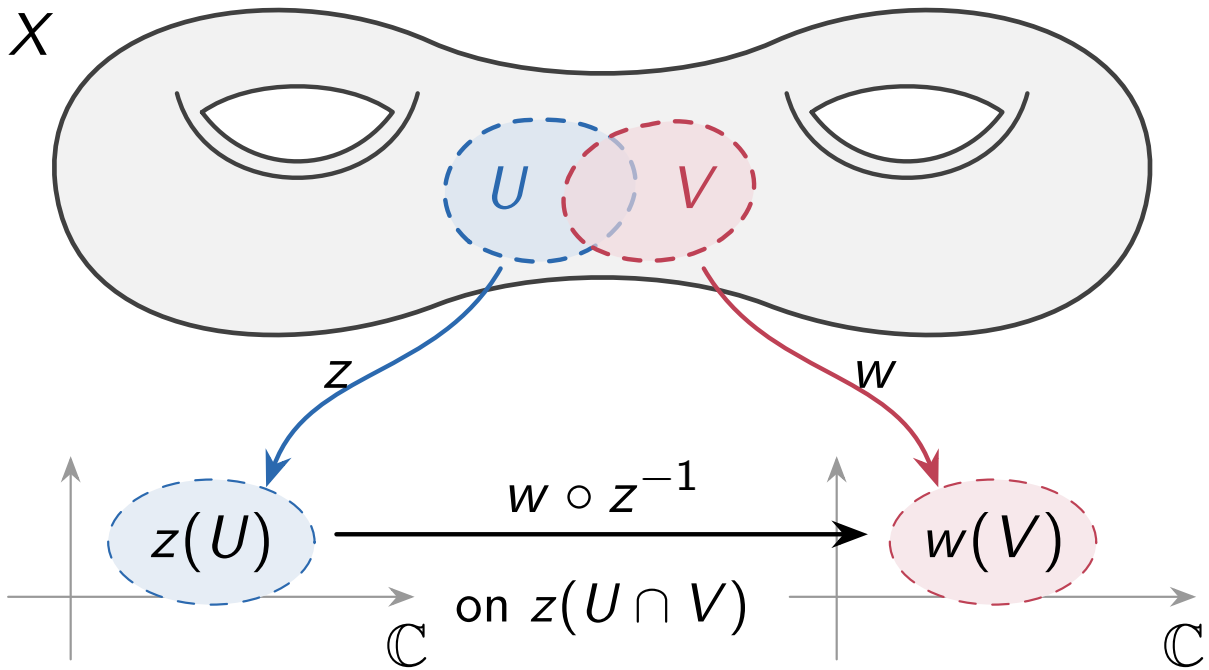
Paul Apisa

University of Toronto

Part 1: The players

Riemann surfaces

A *Riemann surface* is a surface with complex charts and holomorphic transition maps.



Holomorphic 1-forms

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Proof Idea: Take the LES associated to

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X \longrightarrow 0.$$

Strata of holomorphic 1-forms

Theorem (Riemann's Degree Theorem)

On a genus g RS X, every nonzero holomorphic 1-form ω satisfies

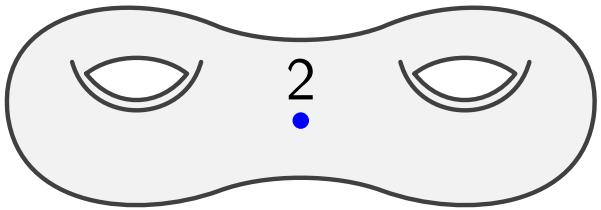
$$\sum_{p \in X} \mathrm{ord}_p(\omega) = 2g - 2.$$

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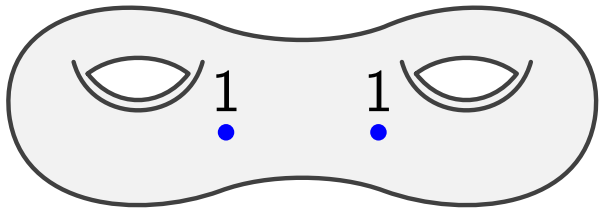
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$\mathcal{H}(2)$



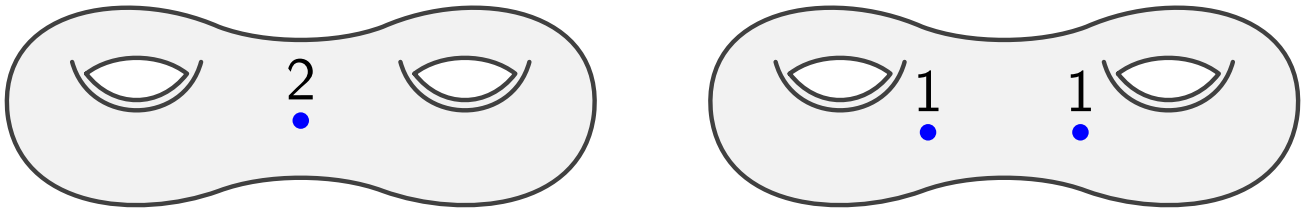
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For $g \geq 2$ and a positive integer partition $\kappa = (k_1, \ldots, k_n)$ of $2g-2$,

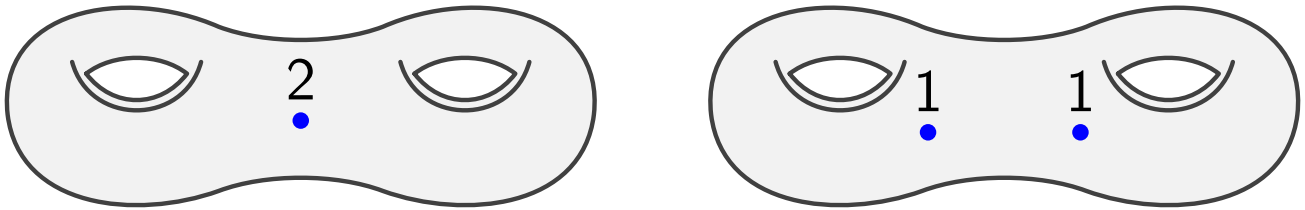
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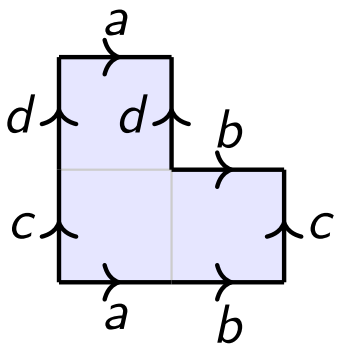
Strata are orbifolds of dim. $2g + n - 1$.

Translation surfaces

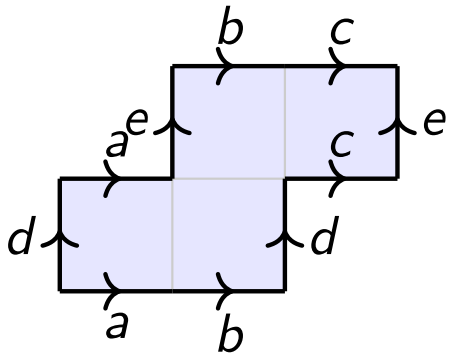
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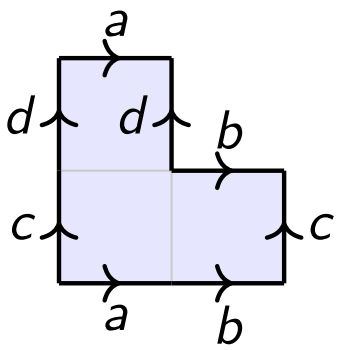
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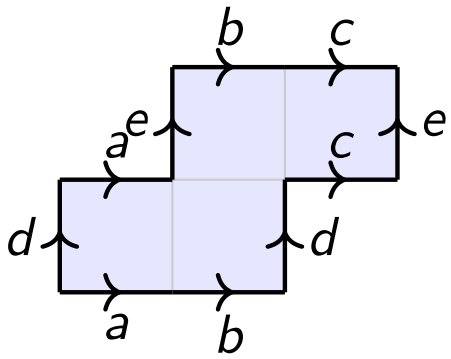
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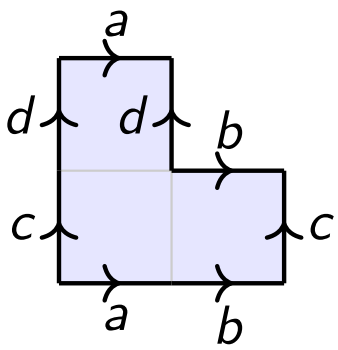
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Theorem

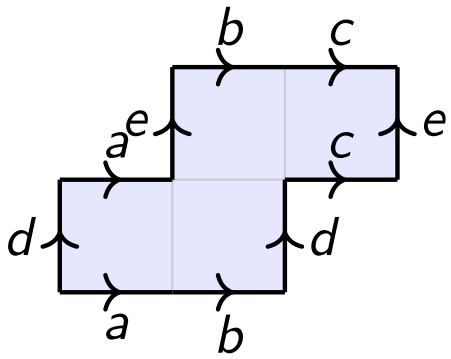
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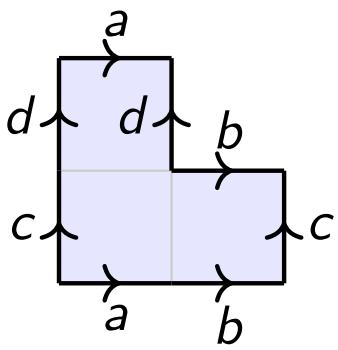
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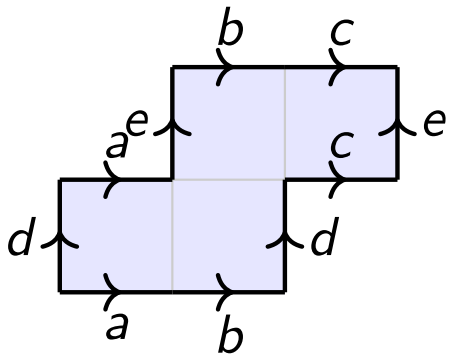
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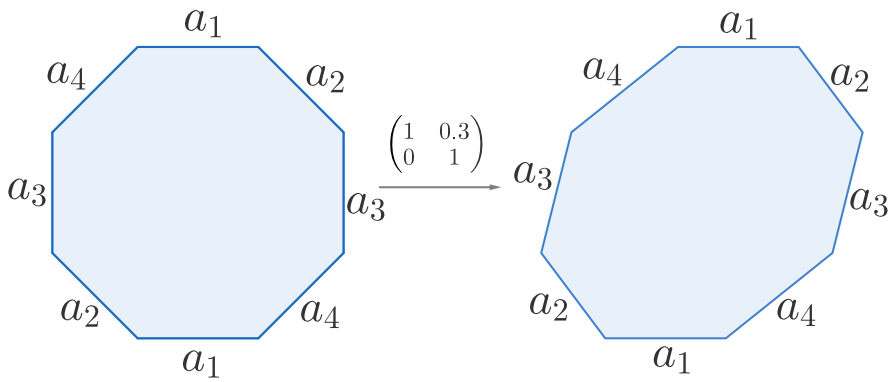
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Strata parameterize trans. surf. with a fixed number of cone points and given cone angles. They are topologized by *period coordinates*, i.e. giving sufficiently many edges of polygons as elements of $\mathbb{C}$.

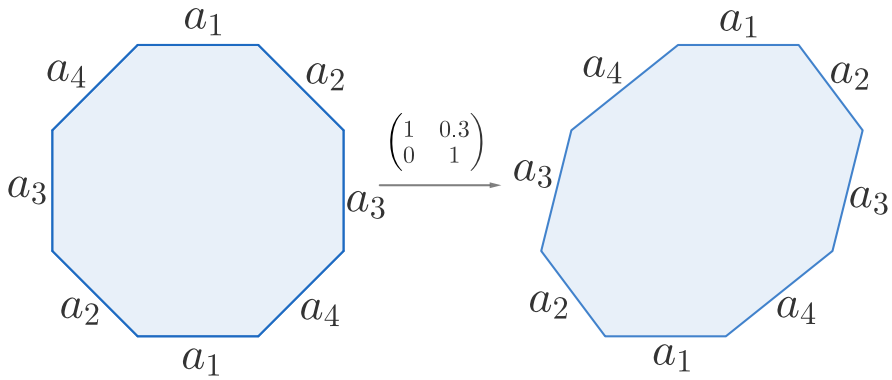
Dynamics on the Hodge Bundle

$\mathrm{GL}(2,\mathbb{R})$ acts on polygons; so it acts on strata.



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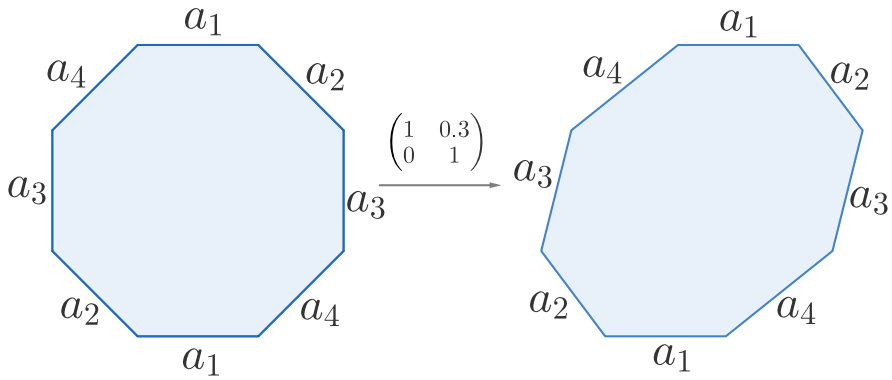
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Rotation by θ fixes the Riemann surfaces and scales the form by $e^{i\theta}$.

Affine invariant subvarieties

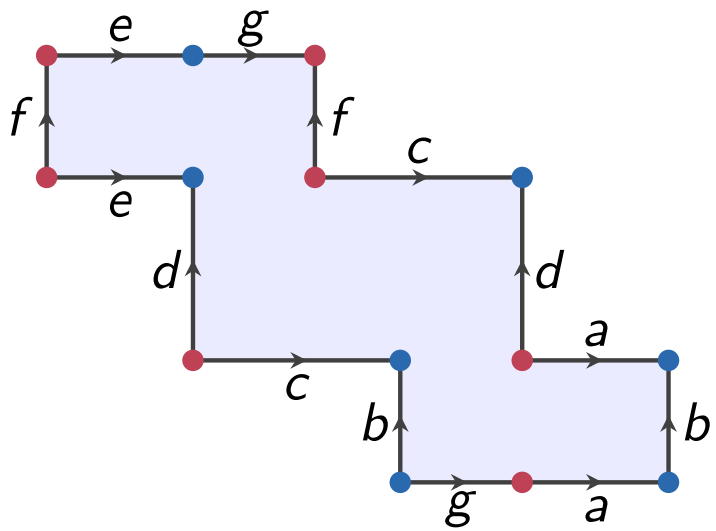
Definition

An *affine invariant subvariety* (AIS) is a closed subset of a stratum defined locally by real linear equations in period coordinates.

Affine invariant subvarieties

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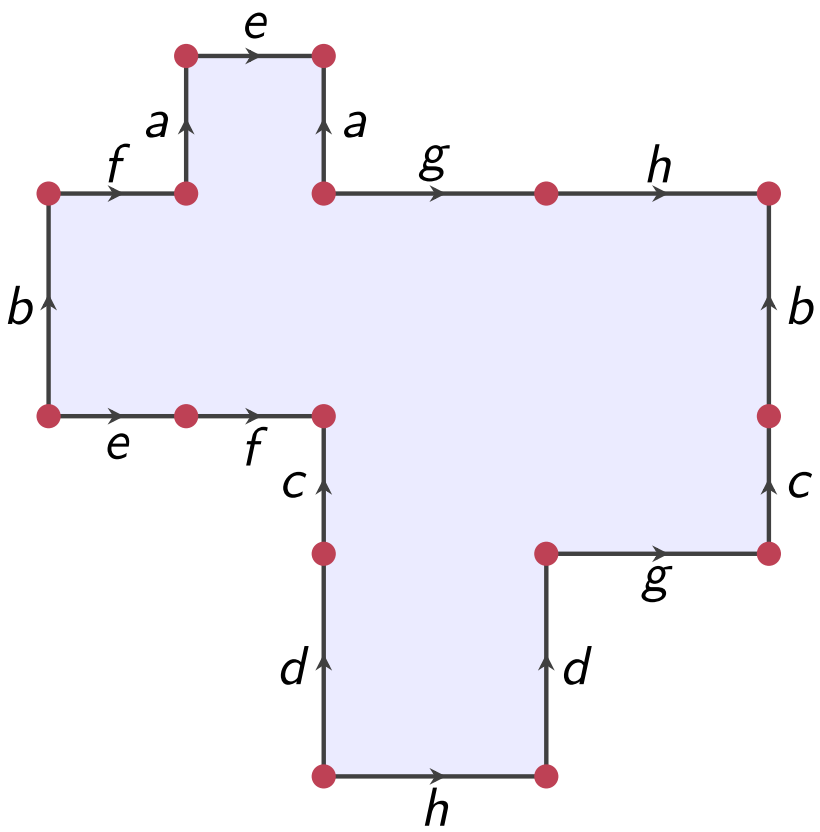
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The Prym locus in $\mathcal{H}(2,2)$.

$$\boxed{\begin{aligned} e &= a, \\ f &= b. \end{aligned}}$$

The EMMW locus in $\mathcal{H}(6)$



$$\phi = \frac{1+\sqrt{5}}{2}.$$

$$\boxed{\begin{aligned} d &= \phi a, \\ b &= \phi c, \\ g &= \phi f, \\ h &= \phi e. \end{aligned}}$$

Why to care about AIS

Theorem (Eskin-Mirzakhani; EM+Mohammadi)

The $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any point in a stratum is an AIS.

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The $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any point in a stratum is an AIS.
The $\mathrm{SL}(2,\mathbb{R})$-invariant finite ergodic measures on (the unit area locus) of strata are (restrictions) of Lebesgue measure on an AIS.

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3. AIS correspond to $\mathrm{GL}(2,\mathbb{R})$-orbit closures and invariant measures.
4. Can we classify them?!?!

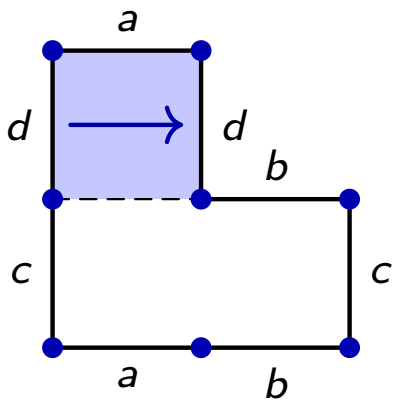
Part 2: Cylinders, Rank, and Rel

Cylinders, shears, and dilations

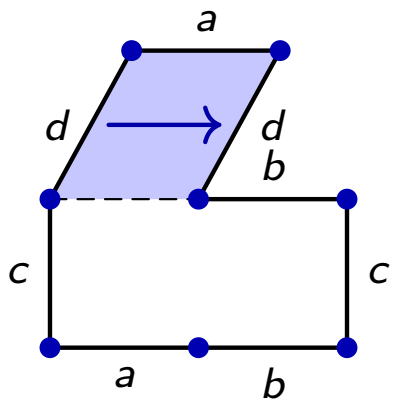
A *cylinder* on a translation surface is an isometrically embedded Euclidean cylinder

$$C \cong (\mathbb{R}/w\mathbb{Z}) \times (0, h),$$

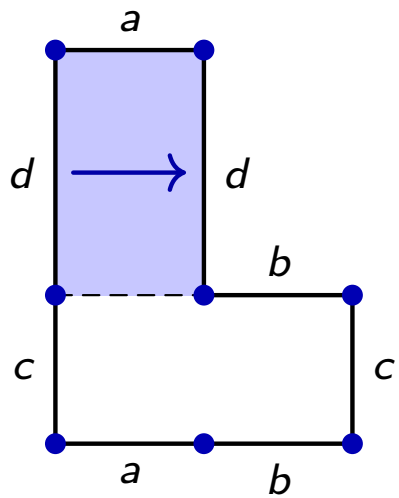
where w is its *circumference* and h is its *height*.



Original surface in $\mathcal{H}(2)$



Shear: height stays 1



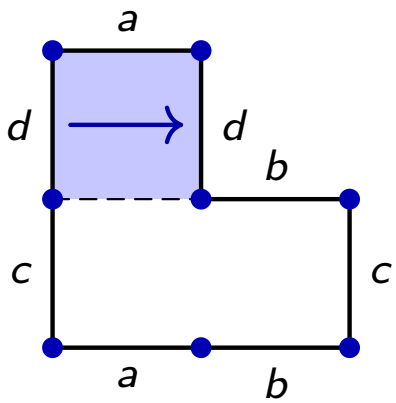
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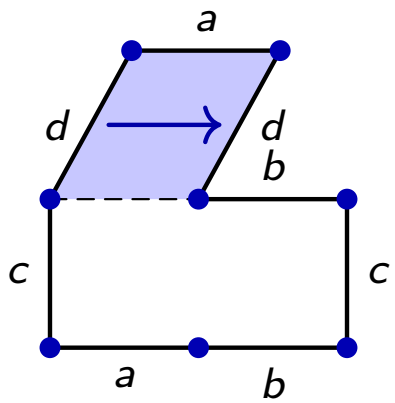
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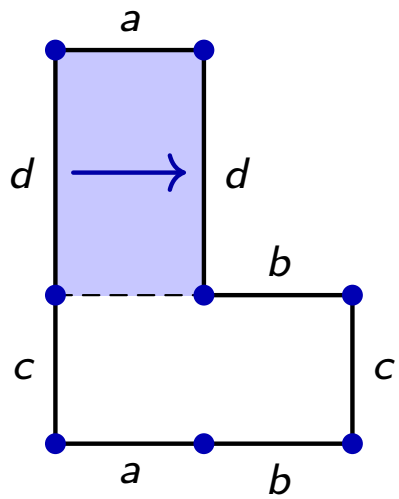
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Cylinder deformations are PL maps of surfaces that act as the identity outside the cylinder. Examples include *cylinder shears* and *cylinder dilations*.

Tangent vectors and cylinder dilation

Given a holomorphic 1-form ω on a Riemann surface X in a stratum $\mathcal{H}$ with zeros Z,

$$T_{(X,\omega)}\mathcal{H} \cong H^1(X, Z; \mathbb{C}).$$

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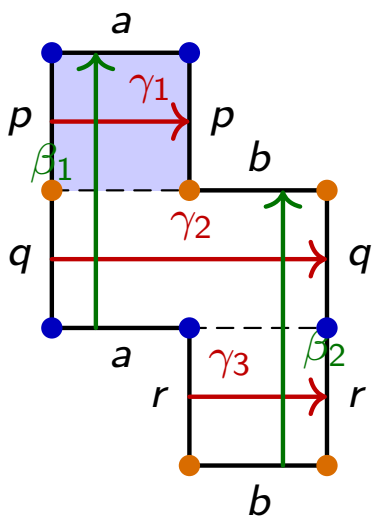
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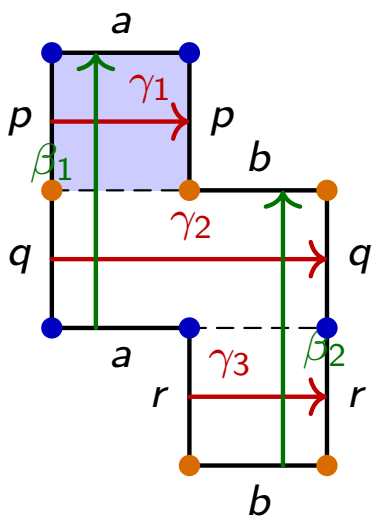


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Using the letters for lengths

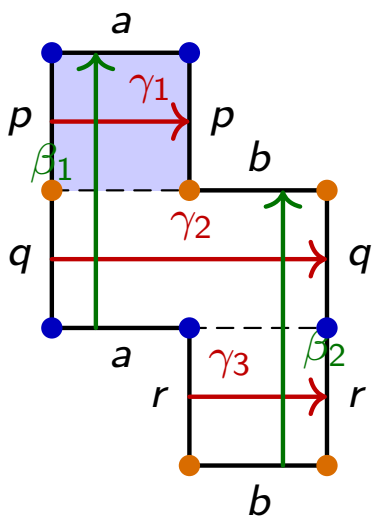
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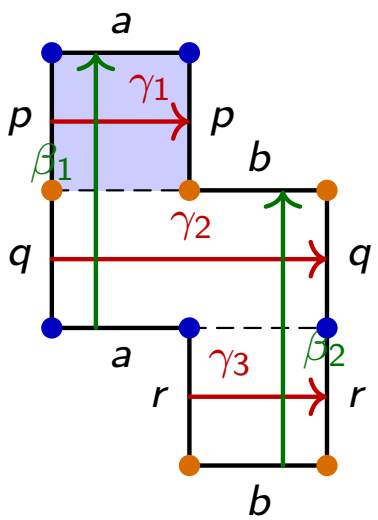
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General principle: If a horizontal cylinder has circumference curve γ and height h, then $ih\gamma^*$ and $h\gamma^*$ are the tangent vectors of the standard dilation and shear resp.

Cylinder equivalence classes

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Lemma

If γ and η are core curves of equivalent cylinders, then, for some $c \in \mathbb{R}$, the equation $\gamma = c\eta$ holds on their periods in $\mathcal{M}$.

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$$\frac{\gamma}{\eta} : U \longrightarrow \mathbb{C}$$

is holomorphic and real-valued, hence constant. □

Cylinder Deformation Theorem

For a horizontal equivalence class $\mathcal{C} = \{C_1, \ldots, C_k\}$, with heights h_i and rightward core curves γ_i, its *standard shear* is

$$\boxed{\sigma_{\mathcal{C}} = \sum_{i=1}^{k} h_i \gamma_i^*.}$$

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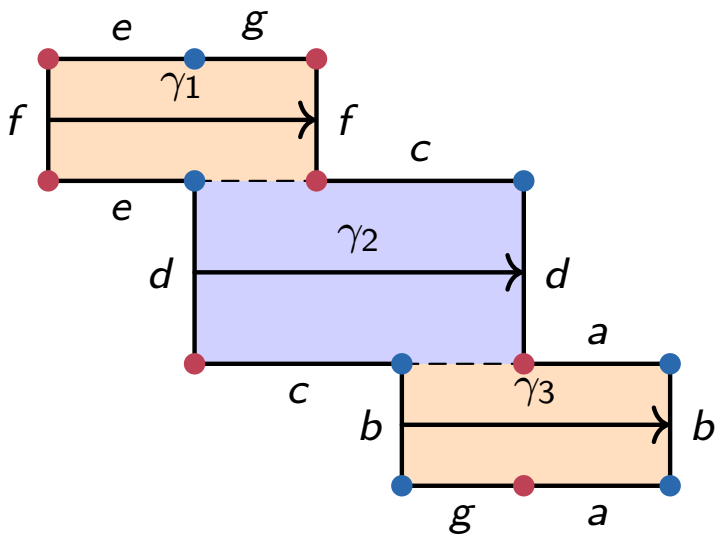
Theorem (Wright's Cylinder Deformation Theorem)

$\sigma_{\mathcal{C}}$ is in the tangent space of $\mathcal{M}$.

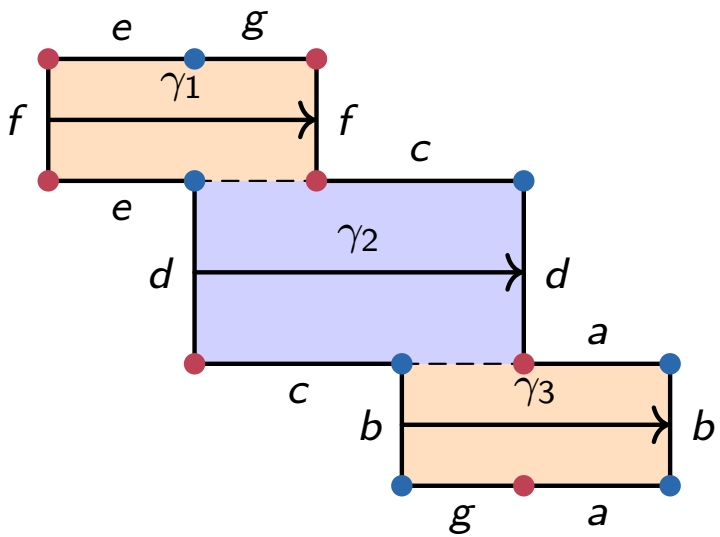
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Periods of horizontal curves are $e + g = a + g$ and $c + g$. So equivalence classes are

$$\mathcal{A} = \{C_1, C_3\}, \qquad \mathcal{B} = \{C_2\}.$$

Standard shears are

$$\boxed{\begin{aligned} \sigma_{\mathcal{A}} &= \gamma_1^* + \gamma_3^*, \\ \sigma_{\mathcal{B}} &= \gamma_2^*. \end{aligned}}$$

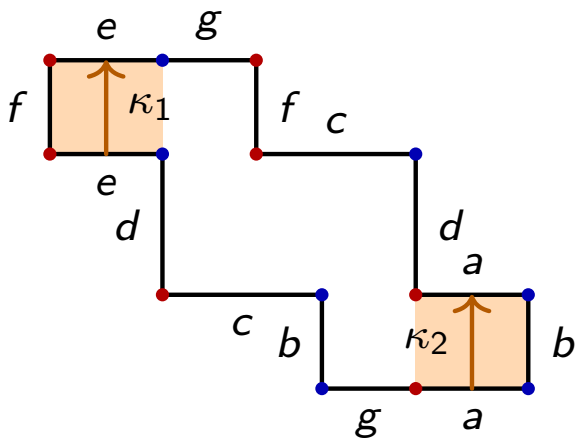
Are these e.c. for the ambient stratum?

Three more cylinder equivalence classes

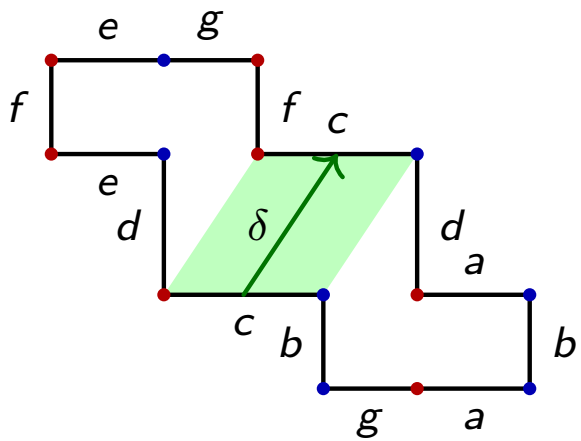
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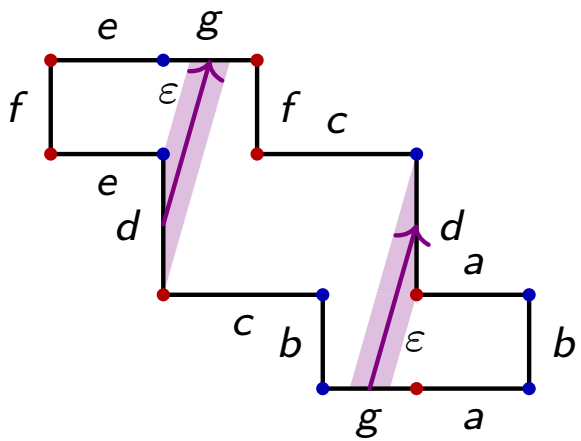
$\mathcal{C}$



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$\mathcal{E}$



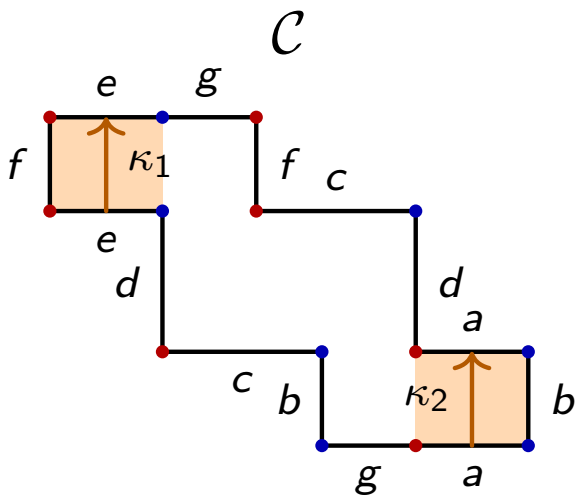
$$\sigma_{\mathcal{C}} = \kappa_1^* + \kappa_2^* \qquad \sigma_{\mathcal{D}} = \delta^* \qquad \sigma_{\mathcal{E}} = \varepsilon^*$$

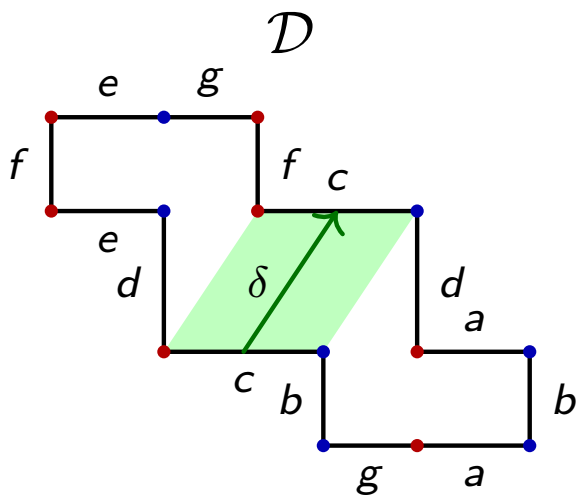
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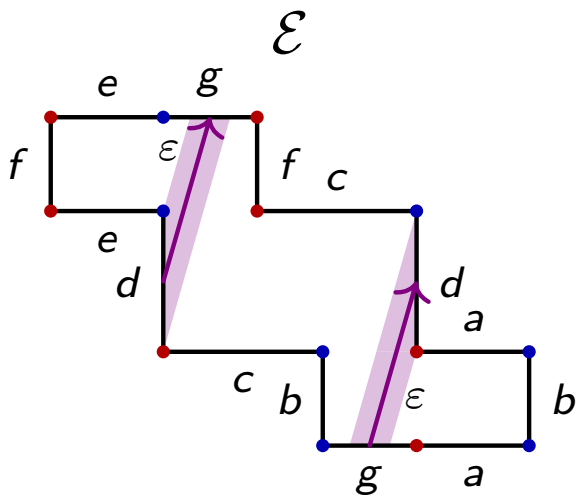
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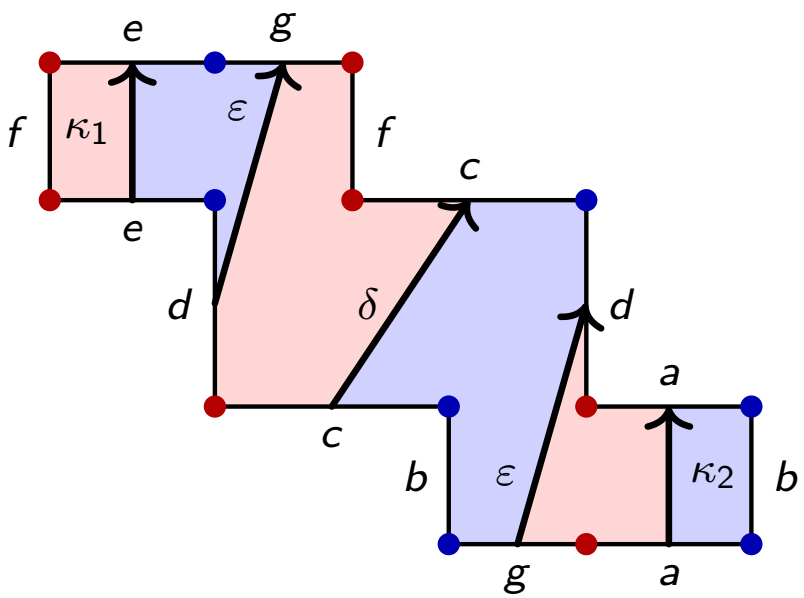


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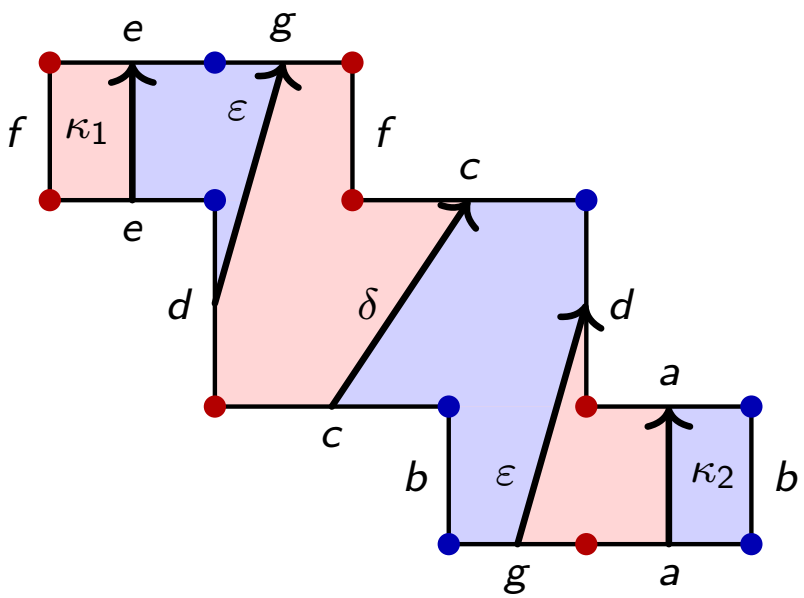
The Prym locus is 5D. These five standard shears span its tangent space.

Rel Vectors



$v := (\kappa_1 + \kappa_2) - \epsilon - \delta$ is homologous to zero since these curves disconnects.

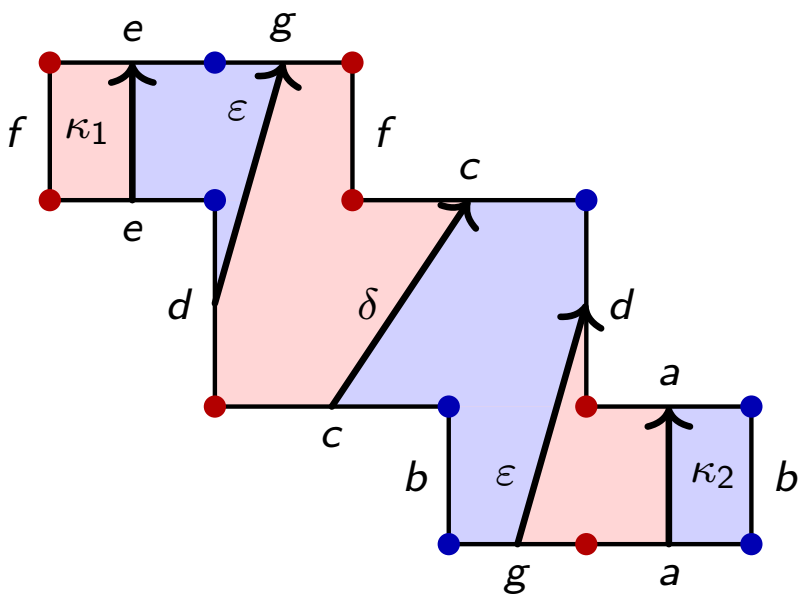
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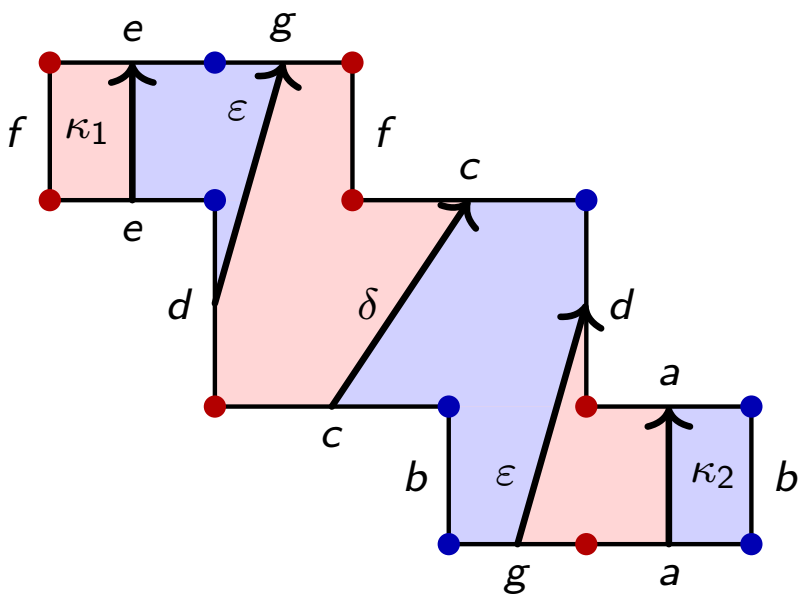


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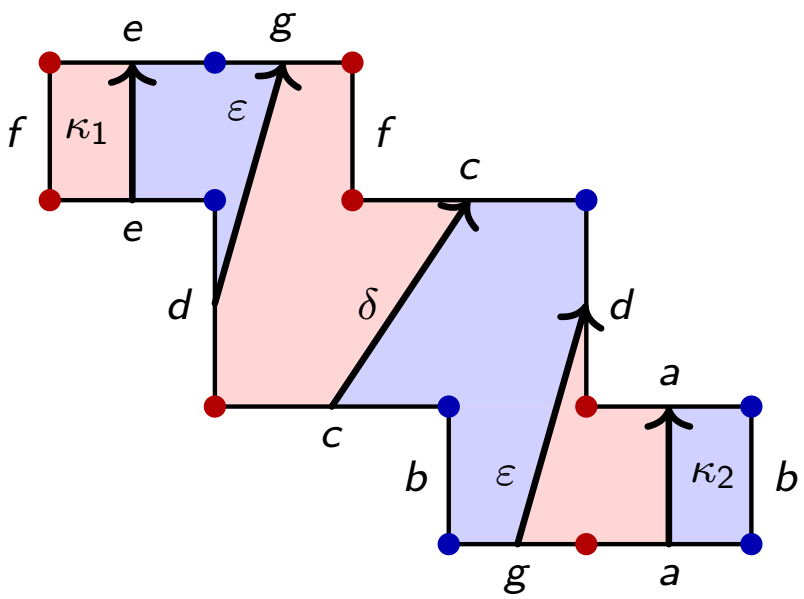


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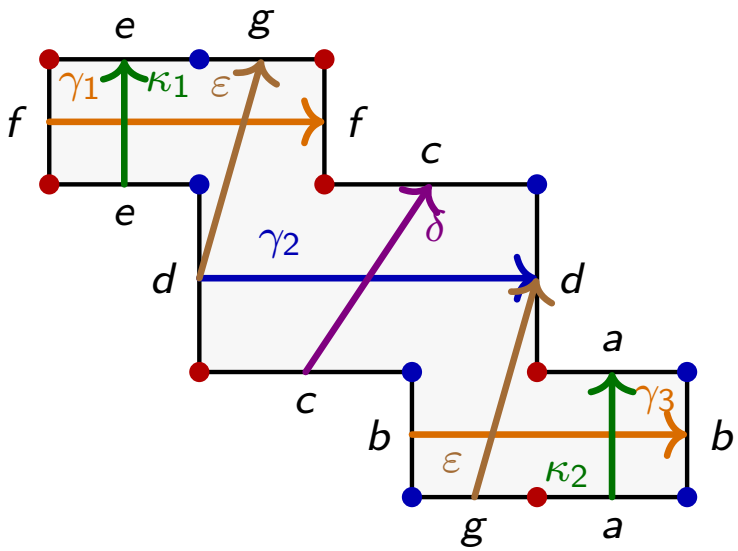


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Rank and Rel



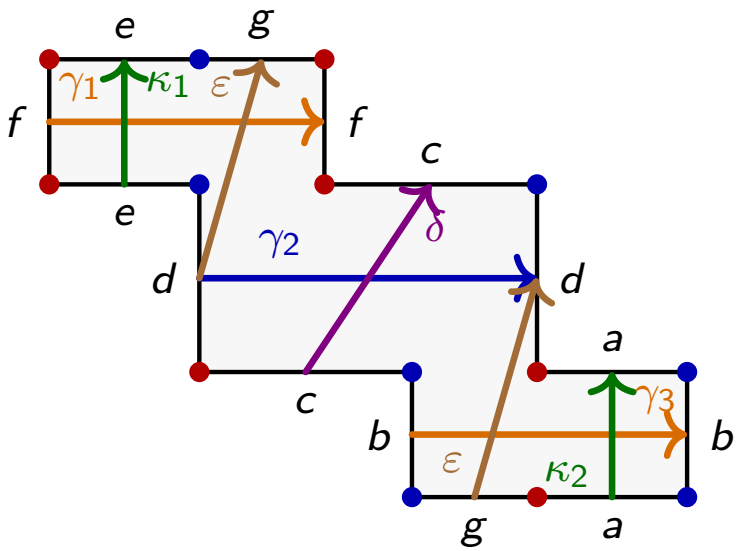
Theorem (Avila–Eskin–Möller)

$p(T\mathcal{M})$ is symplectic.

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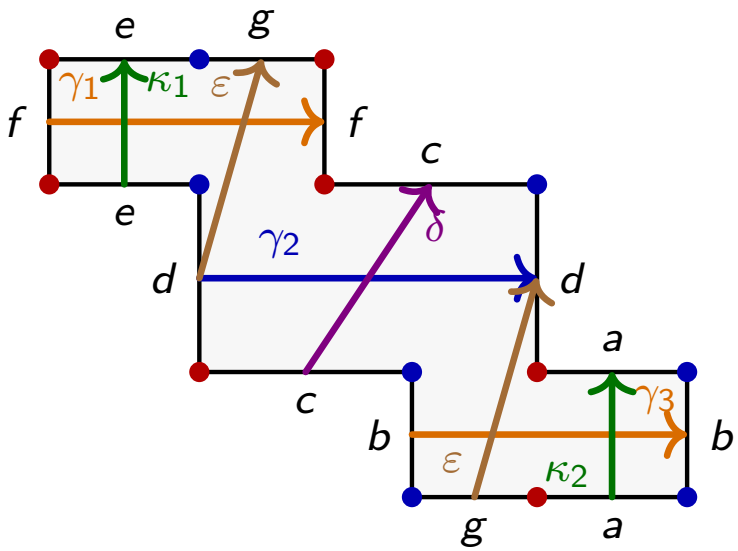
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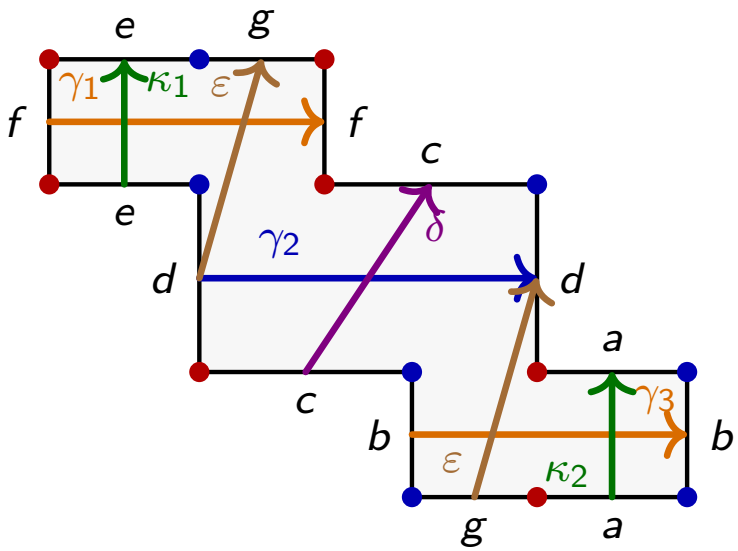
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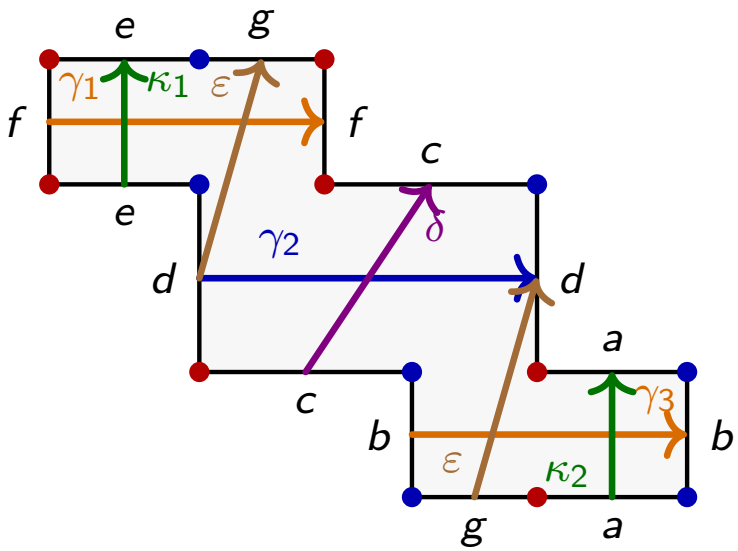
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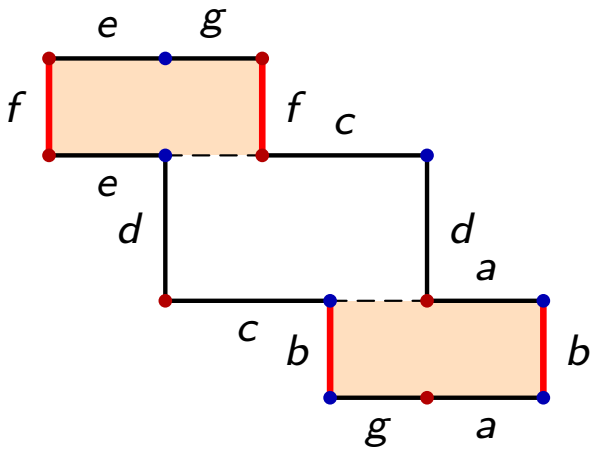
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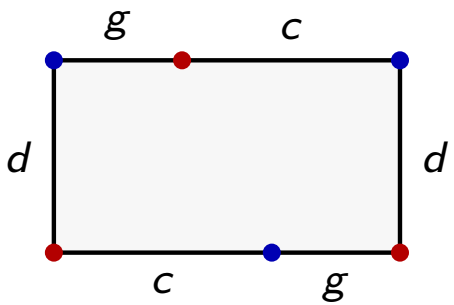
Part 3: The boundary

WYSIWYG: collapsing $\mathcal{A}$



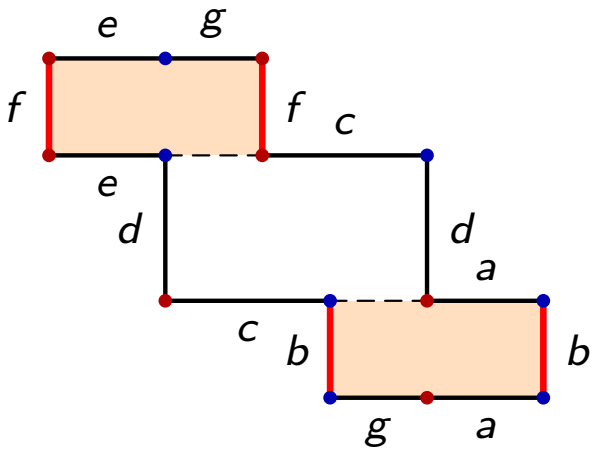
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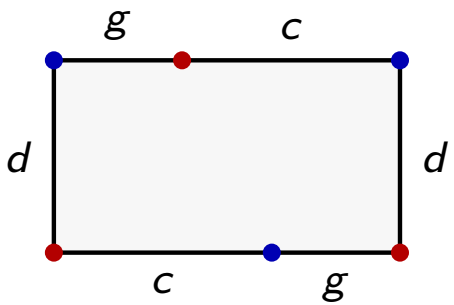


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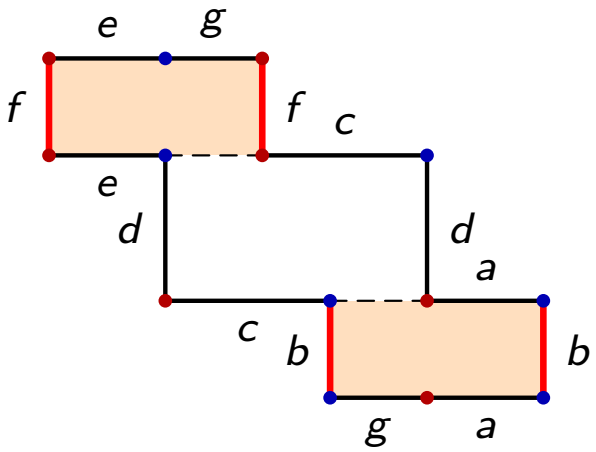


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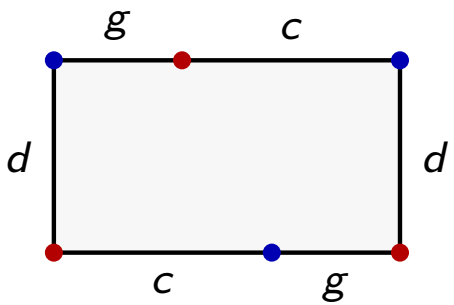
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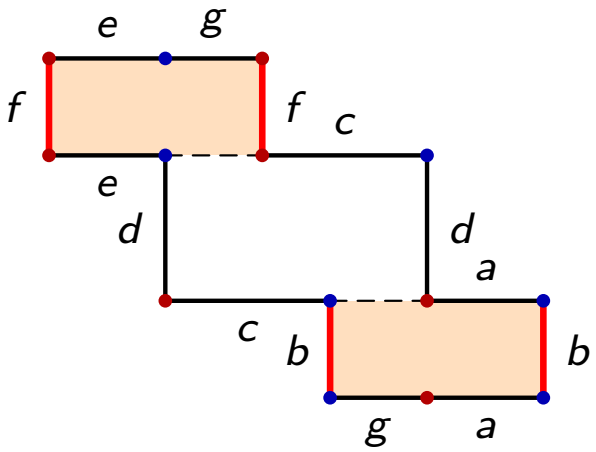
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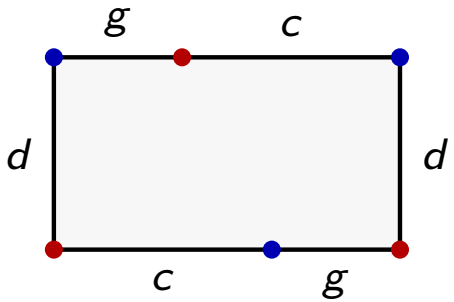
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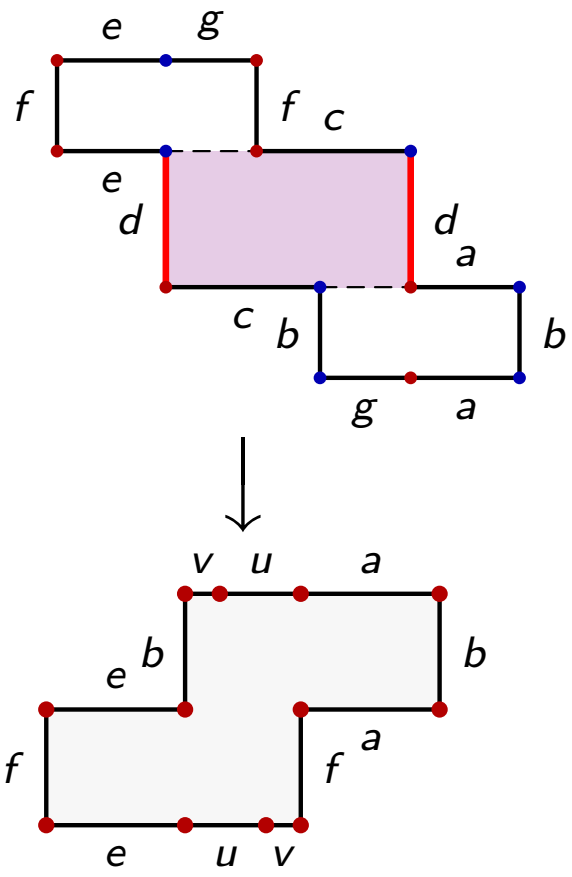
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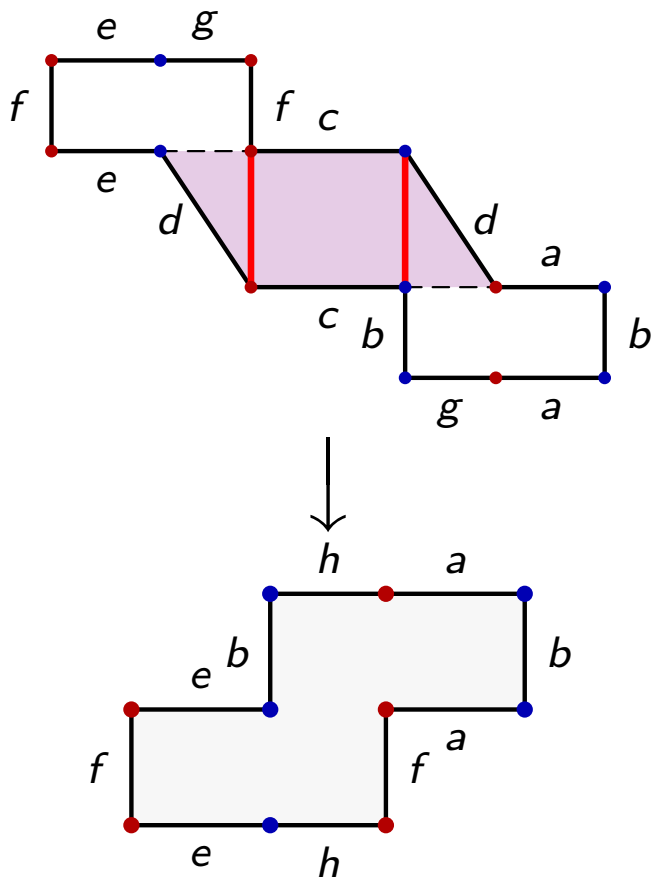
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What are they here?

Collapsing $\mathcal{B}$



$\mathrm{Col}_{\mathcal{B}}(X) \in \mathcal{H}(4)$



$\mathrm{Col}_{\mathcal{B}}(X) \in \mathcal{H}(1,1)$

The Boundary Formula

Let $X' := \mathrm{Col}_{\mathcal{A}}(X)$ and let $f : X \to X'$ be the collapse map. Let $V \subseteq H_1(X, Z)$ be the vanishing. cycles. Let $\mathrm{Ann}(V)$ be elements of $H^1(X, Z)$ that evaluate to zero on V.

Theorem (Mirzakhani-Wright Boundary Formula)

There is an AIS $\mathcal{M}_{\mathcal{C}}$ containing X' where $f^* T_{X'} \mathcal{M}_{\mathcal{C}} = T_X \mathcal{M} \cap \mathrm{Ann}(V)$.

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The Boundary Formula

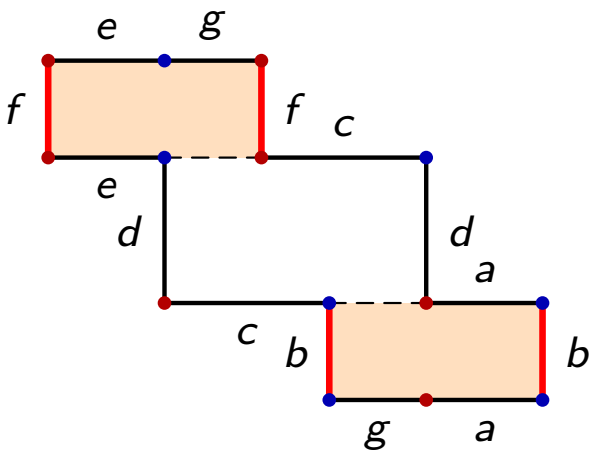
Let $X' := \mathrm{Col}_{\mathcal{A}}(X)$ and let $f : X \to X'$ be the collapse map. Let $V \subseteq H_1(X, Z)$ be the vanishing. cycles. Let $\mathrm{Ann}(V)$ be elements of $H^1(X, Z)$ that evaluate to zero on V.

Theorem (Mirzakhani-Wright Boundary Formula)

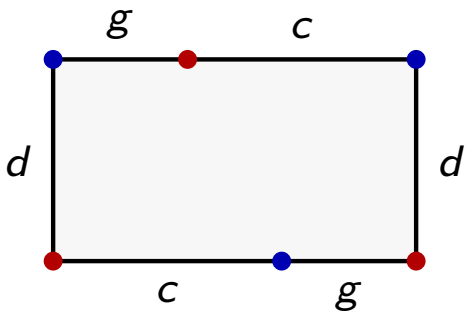
There is an AIS $\mathcal{M}_{\mathcal{C}}$ containing X' where $f^* T_{X'} \mathcal{M}_{\mathcal{C}} = T_X \mathcal{M} \cap \mathrm{Ann}(V)$.

Exercise (The equations defining the boundary): Write out a basis for the vanishing cycles. Extend it to a basis of $H^1(X, Z; \mathbb{Z})$. Write out the equations for $\mathcal{M}$ in the base. Replace each instance of a vanishing cycle with 0.

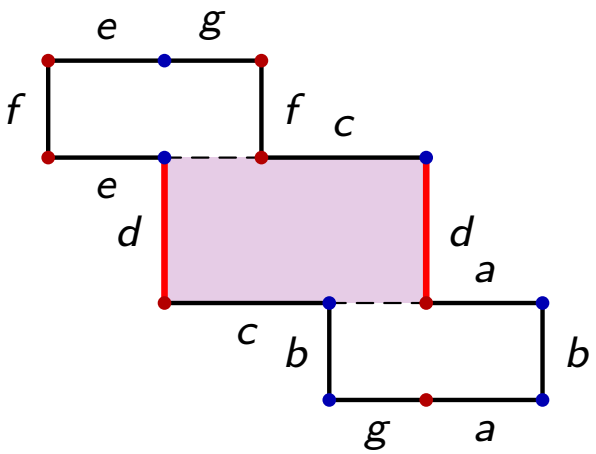
Exercise: The boundary of an AIS always has a smaller dimension and nonincreasing rank. However, rel may increase!



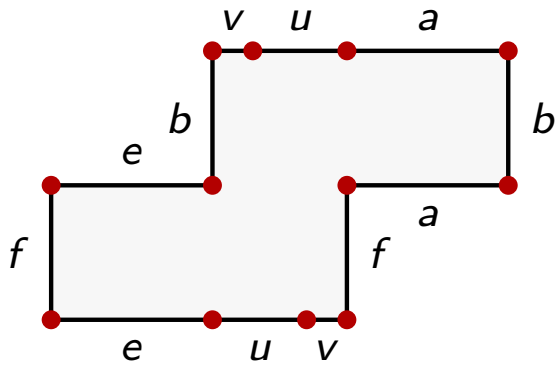
$e = a, \qquad f = b$



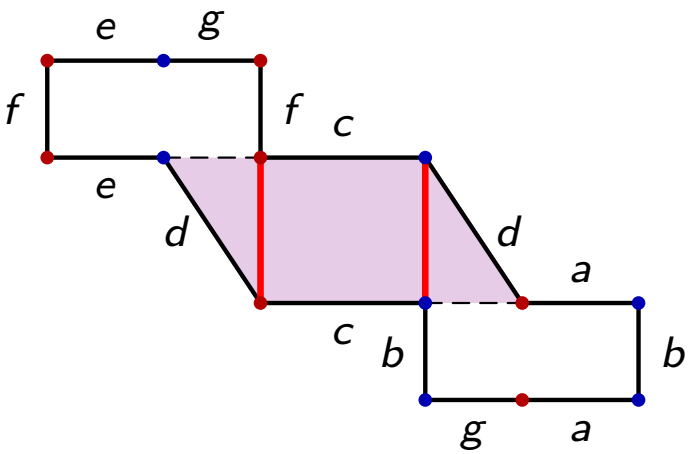
$\mathcal{M}_{\mathcal{A}} = \mathcal{H}(0,0)$



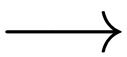
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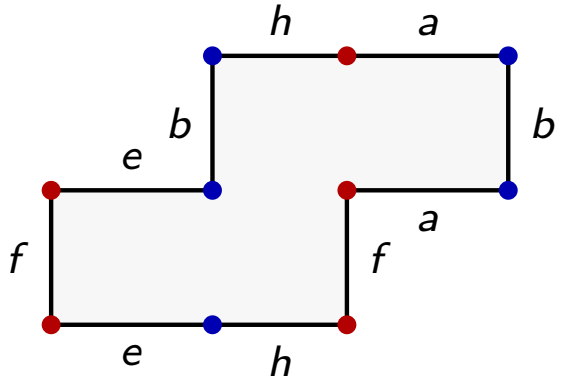


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Part 4: An application

Hyperelliptic Components

Components of strata in which every translation surface has a 180-degree rotational symmetry are called *hyperelliptic*.

Hyperelliptic Components

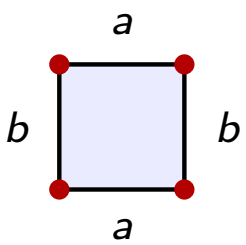
Components of strata in which every translation surface has a 180-degree rotational symmetry are called *hyperelliptic*.

There is one such component in $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1,g-1)$ for all $g \geq 1$.

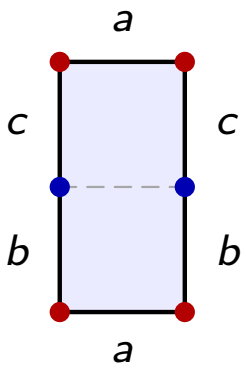
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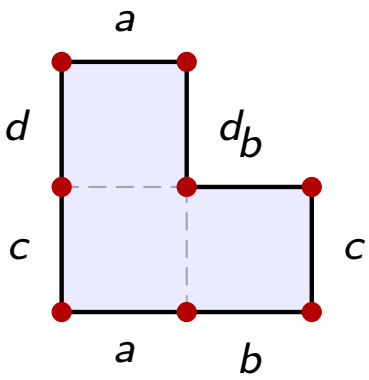
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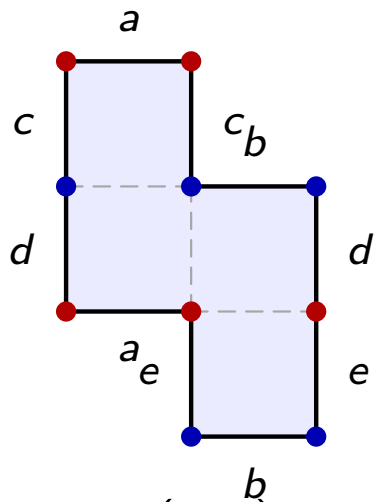
$\mathcal{H}(0)$



$\mathcal{H}(0,0)$



$\mathcal{H}(2)$



$\mathcal{H}(1,1)$

Cyl. degenerations of surfaces in these components still lie in hyperelliptic comp.

Two tools: Tool 1

Lemma

For any rank 2, rel 0 AIS, there is a surface that is horizontally and vertically periodic, with two vertical equivalence classes $\mathcal{A}_1, \mathcal{A}_2$ and two horizontal equivalence classes $\mathcal{B}_1, \mathcal{B}_2$, such that

$$\mathcal{A}_1 \subseteq \mathcal{B}_1, \qquad \mathcal{B}_2 \subseteq \mathcal{A}_2.$$

This is called a *prototype*.

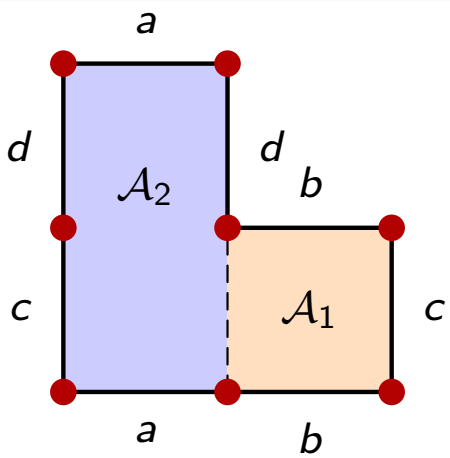
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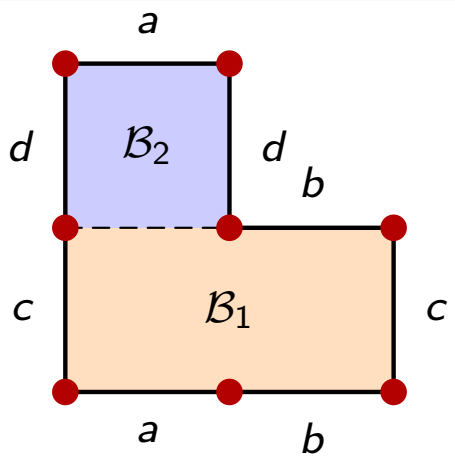
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$$\mathcal{M} = \mathcal{H}(2), \qquad \mathcal{A}_1 \subseteq \mathcal{B}_1, \qquad \mathcal{B}_2 \subseteq \mathcal{A}_2.$$

Two tools: Tool 2

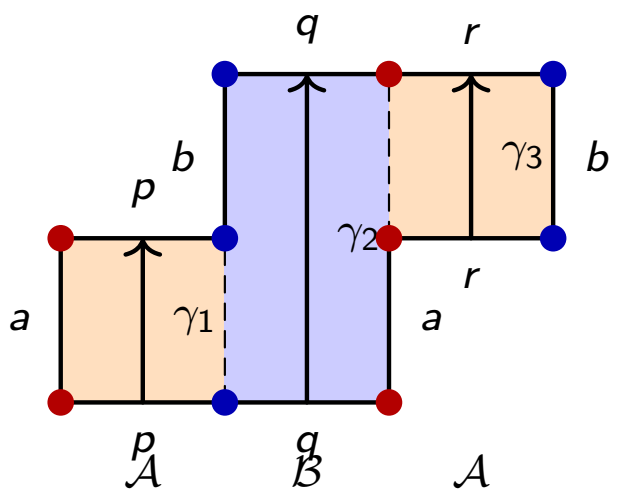
Lemma

Suppose a vertically periodic surface in $\mathcal{H}(1,1)$ has a nonzero rel vector that is a linear combination of the standard shears of a partition $\mathcal{A} \sqcup \mathcal{B}$ of its vertical cylinders. Then, up to interchanging $\mathcal{A}$ and $\mathcal{B}$, its cylinder diagram is the one below.

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In this example:

$$\sigma_{\mathcal{A}} = (\gamma_1^* + \gamma_3^*),$$
$$\sigma_{\mathcal{B}} = \gamma_2^*.$$

$$\gamma_1 + \gamma_3 = \gamma_2 \quad \text{in } H_1(X;\mathbb{Z}).$$

$$\boxed{0 \neq \sigma_{\mathcal{A}} - \sigma_{\mathcal{B}} \in \ker p.}$$

Application: orbit closures in $\mathcal{H}^{\mathrm{hyp}}(4)$

Theorem (Nguyen–Wright)

Every translation surface in $\mathcal{H}^{\mathrm{hyp}}(4)$ has either a closed or a dense $\mathrm{GL}^+(2,\mathbb{R})$-orbit.

Proof: The minimal stratum has no rel (i.e. $p : H^1(X, pt) \cong H^1(X)$).

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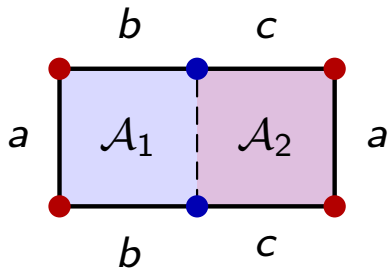
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The relevant hyperelliptic boundary strata are

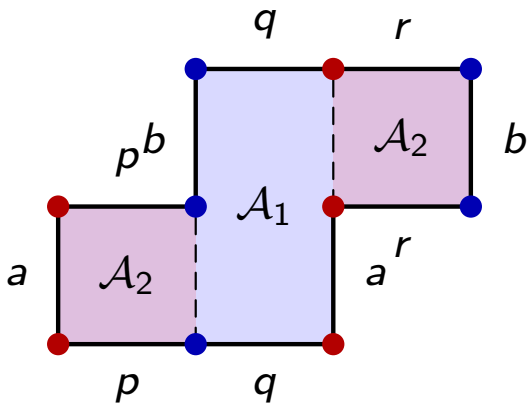
$$\mathcal{H}(0,0) \qquad \text{and} \qquad \mathcal{H}(1,1).$$

The three boundary configurations



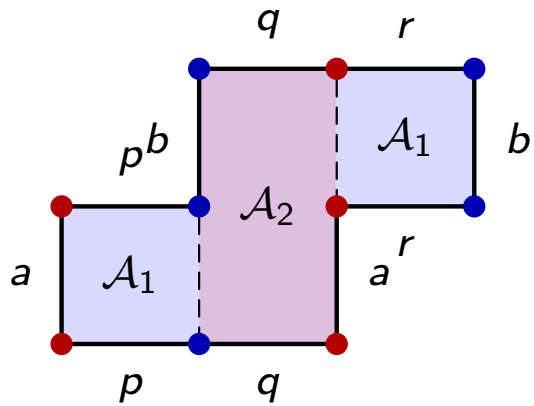
$\mathcal{H}(0,0)$

$\mathcal{A}_2$: one simple cylinder.



$\mathcal{H}(1,1)$

$\mathcal{A}_2$: two simple cylinders.



$\mathcal{H}(1,1)$

$\mathcal{A}_2$: one nonsimple cylinder.

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So by symmetry $\mathcal{A}_1$ and $\mathcal{B}_2$ must both be simple cylinders so that collapsing them takes us to $\mathcal{H}(0,0)$. Collapsing them loses one saddle connection. By hyperellipticity, collapsing goes to $\mathcal{H}(1,1)$, a contradiction.