Real analytic geometry and singularity theory

in honour of the 80th birthday of Edward Bierstone

Abstracts

Janusz Adamus

Monday, 9:00–9:50

Arc-analytic subanalytic functions on complex manifolds

It is well known that an arc-analytic subanalytic function on a real analytic manifold M is real analytic outside a closed subanalytic subset of M, of codimension at least 2. However, the global behaviour of these functions is rather poorly understood. By contrast, when defined over complex manifolds, the local regularity of such functions has profound global consequences. We will show that an arc-analytic subanalytic function on a complex manifold, which is holomorphic near one point, is a holomorphic function on the whole manifold.

François Bernard

Monday, 11:30–12:20

Hölder meromorphic functions in equisingularity theory

A classical problem in singularity theory is to study analytic families of singularities and determine under what conditions their fibers remain similar to one another. Depending on how this similarity is understood, different notions of equisingularity arise, including topological triviality, Whitney equisingularity, and Lipschitz equisaturation. For reduced complex curves, these distinct notions can be characterized by the constancy of certain numerical invariants. In a joint work with Antoni Rangachev, we introduce the notion of Hölder saturation, generalizing the Lipschitz saturation of Pham and Teissier, with the aim of unifying these notions within a common algebraic framework. Using rational powers of a diagonal ideal, we construct a coherent subalgebra of the normalization sheaf whose sections are precisely the locally α-Hölder meromorphic functions. This algebraic description allows us to show that, for any complex analytic germ, there are only finitely many rational numbers $\alpha \in (0, 1]$ that determine all its α-Hölder saturations. Applied to families, this construction gives rise to a notion of α-equisaturation, recovering classical Lipschitz equisaturation when $\alpha = 1$. We will present results for reduced complex curves that relate the different classical notions of equisingularity to α-equisaturation for suitable values of α, with bounds expressed in terms of numerical invariants.

Gal Binyamini

Monday, 10:30–11:20

Point counting: not just for number theorists anymore

In the last couple of decades o-minimality, and the Pila–Wilkie theorem in particular, have had a big impact on Diophantine geometry, leading to proofs of many "unlikely intersection" type questions. In these applications we normally work with universal covering maps of (mixed) Shimura varieties, and the relevant sets are definable in the structure $\mathbb{R}_{\text{an,exp}}$.

I'll review this background, and then discuss some classical "unlikely intersection" problems in different areas such as harmonic and spectral analysis. We'll see that many of the ideas used in the Diophantine context can be transferred to these different contexts, leading to a rich analogy. One interesting feature of the new applications is that they require larger o-minimal structures, and I'll mention how Ed Bierstone's work contributed to this area.

Georges Comte

Wednesday, 11:30–12:20

Motivic curvatures, Vitushkin invariants and metric entropy

I will explain how, in a joint work with I. Halupczok, using motivic integration theory and the notion of riso-triviality, we introduce two new objects in the framework of definable nonarchimedean geometry. The first one is a convenient partial preorder on the set of constructible motivic functions, and the second one is an invariant, denoted V_0, which is a nonarchimedean substitute for the number of connected components. Based on V_0, we obtain the existence of higher nonarchimedean substitutes of real measure geometric invariants V_i, called the Vitushkin variations. We then establish the nonarchimedean counterpart of a real inequality involving the metric entropy and our invariants V_i.

Charles Fefferman

Tuesday, 16:30–17:20 (online)

Fitting smooth functions to data: Unsolved problems

The talk presents several open problems I've met over the last 25 years, in a journey arising from work of Bierstone, Milman and Pawłucki.

Lise Fonteyne

Friday, 9:00–9:50

On a generalized monodromy conjecture for curves using differential forms

Motivic and topological zeta functions are singularity invariants, mainly associated to a function f and a top differential form ω on a smooth variety. When ω is the standard form $dx_1 \wedge \cdots \wedge dx_n$ on affine n-space, the monodromy conjecture states that poles of these zeta functions should induce monodromy eigenvalues of f. In this talk, I will provide an overview of the zeta functions and the monodromy conjecture. Next, I will introduce generalized zeta functions for functions f on complex surface germs; more precisely on singular surfaces for forms ω that generalize the standard form, and on the affine plane for forms ω that are intrinsically associated to f. This is joint work with Wim Veys.

Anne Frühbis-Krüger

Friday, 11:30–12:20 (online)

Blowing up and Desingularization using OSCAR

Computing a resolution of singularities of a given example by hand quickly becomes tedious, if it is not one of the simplest, straightforward cases. Bierstone and Milman and independently Villamayor and his collaborators have transformed Hironaka's monumental proof of resolution of singularities into algorithms leading the way to implementation. But even in the old Singular implementation the amount of data arising in the process was hard to handle, because the system did not allow structured data at the time. In the modern computer algebra system OSCAR, a framework for schemes, blowing up and resolution of singularities has been created and is constantly emerging, which facilitates this task significantly. In this talk I will report on this framework and how to apply it for resolving singularities.

Riccardo Ghiloni

Thursday, 9:00–9:50

The Nash–Tognoli theorem over the rationals and its version for isolated singularities

Let $\mathbb{Q}$ be the field of rational numbers and let V be a subset of $\mathbb{R}^n$. We say that V is $\mathbb{Q}$-*algebraic* if it is the common zero set in $\mathbb{R}^n$ of a family of polynomials in $\mathbb{Q}[x_1, \dots, x_n]$. If V is $\mathbb{Q}$-algebraic and of dimension d, then a point $a \in V$ is called $\mathbb{Q}$-*nonsingular* if there exist a neighborhood U of a in $\mathbb{R}^n$ and $f_1, \dots, f_{n-d} \in \mathbb{Q}[x_1, \dots, x_n]$ such that $\nabla f_1(a), \dots, \nabla f_{n-d}(a)$ are linearly independent and $V \cap U = \{x \in U : f_1(x) = 0, \dots, f_{n-d}(x) = 0\}$. Evidently, a $\mathbb{Q}$-nonsingular point of V is also a nonsingular point of V in the usual real algebraic sense, i.e., the set $\mathrm{Reg}^{\mathbb{Q}}(V)$ of all $\mathbb{Q}$-nonsingular points of V is contained in $\mathrm{Reg}(V)$. This inclusion can be strict: for example, if V is the $\mathbb{Q}$-algebraic curve $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 - \sqrt[3]{2}\, x_1 = 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 - 2x_1^3 = 0\}$ of $\mathbb{R}^2$, then $\mathrm{Reg}^{\mathbb{Q}}(V) = V \setminus \{(0,0)\} \subsetneq V = \mathrm{Reg}(V)$. If $\mathrm{Reg}^{\mathbb{Q}}(V) = V$, then V is called $\mathbb{Q}$-*nonsingular*. If $\mathrm{Reg}^{\mathbb{Q}}(V) = \mathrm{Reg}(V)$, then V is called $\mathbb{Q}$-*determined*.

We improve the Nash–Tognoli theorem as follows:

Theorem A. *Every compact C^∞ manifold X of dimension d is C^∞ diffeomorphic to a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset X' of $\mathbb{R}^{2d+1}$.*

This result guarantees for the first time that, up to C^∞ diffeomorphisms, every compact C^∞ manifold X can be described both globally and locally by means of a matrix of integers provided by the coefficients of a finite system of generators of the $\mathbb{Q}$-vanishing ideal $\{q \in \mathbb{Q}[x_1, \dots, x_{2d+1}] : q(x) = 0, \forall x \in X'\}$ of X' in $\mathbb{Q}[x_1, \dots, x_{2d+1}]$, after clearing denominators. This matrix gives us a 'number plate' identifying the C^∞ diffeomorphic class of X.

Using Theorem A, resolution of singularities and some new $\mathbb{Q}$-algebraic approximation techniques, we prove:

Theorem B. *Let $V \subset \mathbb{R}^n$ be a real algebraic set with isolated singularities. Then there exist $m \geq n$, a real algebraic set with isolated singularities $V' \subset \mathbb{R}^m$ and a semialgebraic homeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ such that:*

(1) *V' is $\mathbb{Q}$-algebraic and $\mathbb{Q}$-determined,*
(2) *$\phi(V) = V'$ and $\phi(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$,*
(3) *$\phi|_{\mathrm{Reg}(V)}$ is a Nash diffeomorphism.*

In this talk, we present Theorems A and B, and provide an overview of their proofs. The talk is based on joint work with Enrico Savi: *The Nash–Tognoli theorem over the rationals and its version for isolated singularities*, arXiv:2302.04142.

Vincent Grandjean

Tuesday, 9:00–9:50

About the gradient non-oscillation conjecture

I would like to discuss the gradient non-oscillation conjecture in dimension three. It is a joint work with Professor Tadeusz Mostowski.

Wojciech Kucharz

Thursday, 16:30–17:20

Algebraically constructible manifolds and maps

Let X be a nonsingular real algebraic set. I will present a characterization of the compact smooth submanifolds of X that can be approximated by constructible submanifolds of X, where a constructible submanifold is a smooth submanifold that is constructible in the Zariski topology. Furthermore, I will discuss results on the approximation of maps with constructible graphs by regular maps and by C^k rational maps, where k is a nonnegative integer.

Olivier Le Gal

Friday, 10:30–11:20

Trajectories of vector fields asymptotic to formal invariant curves

We prove that a formal curve Γ, that is invariant by a smooth vector field X on $\mathbb{R}^m$ has a geometrical realization, this means that there exists a trajectory of X which is asymptotic to Γ — as soon as the Taylor expansion of X is not identically zero along Γ. It answers a question asked by Bonckaert in 1982. This is a joint work with F. Sanz.

Lorena López Hernanz

Tuesday, 11:30–12:20

Logarithmic-power series solutions of vector fields in dimension two

We are interested in the study of formal solutions of a vector field X in $(\mathbb{C}^2, 0)$. Camacho and Sad proved that X always has an invariant curve, which is analytic whenever X is. Cano gave an alternative proof of their result using the Newton–Puiseux polygon of X, showing that any side of the polygon that satisfies a certain condition (which always holds for at least one side) provides a "leading term" of an invariant curve, which is again analytic whenever X is.

We extend Cano's result, showing that almost every side of the polygon provides a leading term of a solution, which in general is not given by a power series but rather by a logarithmic-power series (i.e. a series whose expansion includes powers and logarithms of the variable) and which in general is not analytic even when X is. We will also discuss some possible applications of this result to the study of the local dynamics of tangent-to-the-identity biholomorphisms in $(\mathbb{C}^2, 0)$.

This is a joint work in progress with José Cano and Sergio Carrillo.

María Martín Vega

Wednesday, 10:30–11:20

Invariants of cuspidal singularities

In this talk, we will explore the properties of cuspidal varieties. In particular, when they are embedded on a Riemannian manifold, we will analyze how the induced metric is transformed by a resolution of singularities. We see that the metric is of a particular type that we call cuspidal, and we give normal forms and invariants of them. If time allows, we will study properties of geodesics approaching the singularity on the simpler cuspidal varieties.

This talk is based on an ongoing joint work with André Belotto da Silva and Daniel Grieser.

Adam Parusiński
Monday, 16:30–17:20

Perturbation of polynomials and linear operators

The perturbation of polynomials and linear operators is a classical subject that began with Rellich's work in the 1930s. The parameter dependence of the polynomials (resp. operators) ranges from real analytic to C^∞ and differentiable of finite order, with drastically different regularity results often observed for their roots (resp. eigenvalues and eigenvectors).

In this talk, I will present several recent results, such as an optimal Sobolev regularity estimate for the roots of monic complex polynomials of one variable with smooth dependence on a single real parameter; multiparameter versions of this result; and the continuity of the coefficients-to-roots map with respect to the C^d and Sobolev norms. Recently, Antonini, Cavalletti and Lerario reinterpreted some of these results in terms of the Wasserstein metric in order to study optimal transport between algebraic hypersurfaces in complex projective space.

(Based on joint work with Armin Rainer.)

Wiesław Pawłucki
Thursday, 18:00–18:50

Scientific achievements of Edward Bierstone

Armin Rainer
Tuesday, 10:30–11:20

Effective quasianalytic Remez inequalities on tame sets and applications

The classical Remez inequality bounds the supremum norm of a polynomial on the unit interval by its norm on a measurable subset E of positive measure, with a constant depending only on the degree and on $|E|$. Analogous inequalities for polynomials are known in higher dimensions on convex bodies and on certain sets with cusps. We establish an effective Remez inequality for functions in quasianalytic Denjoy–Carleman classes C_M on a large family of compact fat sets with tame geometry, including compact uniformly polynomially cuspidal sets and, in particular, compact fat subanalytic sets. The polynomial degree is replaced by the Bang degree, an integer determined by the weight sequence M and the size of the function. This yields quantitative propagation-of-smallness principles for quasianalytic functions. The Bang degree provides a natural complexity parameter: it controls both the Remez constant and the number of zeros. As applications, we derive effective quasianalytic versions of the Łojasiewicz inequality and obtain Harnack- and Markov-type inequalities. All constants are explicit in terms of the Bang degree and the geometry of the underlying set.

Tamara Servi
Thursday, 10:30–11:20

Integral transforms of tame functions

I will give an overview of a series of joint works on parametric integration and parametric integral transforms of certain classes of functions definable in suitable o-minimal structures. Though the initial setting is that of non-oscillatory real functions, the Fourier and Mellin transforms quickly take us out of the realm of o-minimality. Nevertheless, a certain geometric tameness is preserved in the process, in the form of uniform asymptotic behaviour of parametric families and of almost everywhere regularity of the functions produced by transform.

Patrick Speissegger
Wednesday, 9:00–9:50

Multisummability relative to certain quasianalytic classes related to Dulac's problem

Using Tougeron's characterization of multisummable series (in the positive real direction), the latter can be viewed as infinite series of convergent power series with radii of convergence shrinking to 0. In joint work with Jean-Philippe Rolin and Tamara Servi we showed that, if we replace "convergent power series" with "convergent generalized power series", we obtain a larger class of multisummable series (again in the positive real direction). This class is shown to generate an o-minimal expansion of the real field, whose further expansion by the exponential function then defines the restrictions to some unbounded interval of both the Gamma and zeta functions. More recently, with Ilgwon Seo, we have been further generalizing this construction by replacing "convergent generalized power series" with "0-almost regular generalized power series". The resulting Hardy field is a first step towards filling the remaining gap in Ilyashenko's proof of Dulac's problem.

Michael Temkin
Thursday, 11:30–12:20

Dream resolution for excellent schemes

A few years ago it was discovered independently by McQuillan and Abramovich–Temkin–Włodarczyk that one can extend the usual toolbox of embedded resolution by allowing weighted blowups at cost of viewing the output as a smooth orbifold rather than a singular scheme (which is nothing else but the coarse moduli space of this orbifold), and once the toolbox is extended, a much simpler and efficient resolution algorithm naturally emerges. Moreover, this algorithm has no memory and improves the basic invariant at each step (a dream which cannot be realized if one only uses smooth non-weighted centers), and yet each step is determined by the parameters and degrees at the year zero of the "classical" algorithm of Bierstone–Milman and Villamayor. One more bonus of the new method is that it is much easier to transfer it to other settings and McQuillan proved that his method works for arbitrary excellent schemes of characteristic zero.

In my talk I will discuss my two recent preprints on the weighted method. The first suggests a short justification of the method, which works in the generality of schemes with enough derivations, is independent of the usual theory of maximal contacts and shifts the accent to studying the weighted centers. And the second work generalizes the theory to arbitrary excellent schemes and formalizes the notion of a tube associated with relevant weighted centers and may be of an independent interest.

Anna Valette
Monday, 18:00–18:50

In search of definable Whitney Extension Theorem à la Whitney

In 1997, K. Kurdyka and W. Pawłucki proved a subanalytic version of Whitney's extension theorem. Since then, several proofs in the o-minimal setting have been produced. In this talk, I will explain how we can obtain an even simpler proof by using a definable variant of Urysohn's Lemma for class C^q. This is based on recent joint work with B. Kocel-Cynk and W. Pawłucki.
