WORKSHOP: SYMPLECTIC STRUCTURES AND TRISECTIONS

CIRM, 17–21 NOVEMBRE 2025

(1) **Symplectic manifolds**
Give the definition of symplectic manifold, symplectomorphism, Lagrangian submanifold. Examples: dimension 2, cotangent bundles, complex projective spaces. We may focus on dimension 4. It would be especially useful to develop a clear picture of the standard symplectic structure on $\mathbb{CP}^2$. Reference: [CG18, §2.2 and Lemma 2.5].

(2) **Neighborhood theorems**
Present Moser's lemma for symplectic forms, Darboux's theorem for symplectic structures and Weinstein's neighborhood theorem. Reference: [CG18, §2.4].

(3) **Symplectic submanifolds and complex curves, including singularities**
Discuss symplectic surfaces in symplectic 4–manifolds and show that the symplectic structure in a neighborhood of a symplectic surface is determined by the genus, symplectic area and self-intersection of the surface [MS17, Theorem 3.4.10 and Exercise 3.4.11]. Show that smooth algebraic curves in $\mathbb{CP}^2$ are symplectic. Discuss singularities that can arise, and present the discussion of singularities in [LCMS21, §3.1] up to and including the proof of Lemma 3.5.

(4) **Smooth handle decompositions**
Focusing on dimensions up to 4, define handles and handle decompositions. Explain handleslides and cancelling pairs of handles. Sketch the existence proof using Morse functions. Define Heegaard splittings and Heegaard diagrams of closed 3–manifolds. Give a proof of existence. Give examples throughout (including $\mathbb{CP}^2$). References: [GS, Chapter 4],[OS06b, §2&3], [Sc, §1.2–1.3], [Ak, §1.1–1.4].

(5) **Smooth trisections**
Give the definition of trisections (unbalanced), and explain the stabilization moves. Give a proof of existence via Morse functions [LCM21, §2.1]. Define trisection diagrams. Give examples throughout, in particular that of $\mathbb{CP}^2$. References: [GK16], [Gay18], [LC20, §1.2].

(6) **Bridge trisections**
Explain the definition of generalized bridge trisections. Give examples, in particular bridge positions of surfaces in $\mathbb{CP}^2$. If time permits, sketch the existence proof. References: [MZ18, MZ17].

(7) **Bridge trisections of algebraic curves**
Show how to construct natural bridge trisections of algebraic curves in $\mathbb{CP}^2$, as in [LCM20, §4], and in particular show how to get their shadow diagrams. If time permits, discuss destabilizing these to "efficient bridge trisections".

(8) **Branched covers**
Define branched covers in dimension 2 and give examples (see *eg* [Gol, §2.3]). Then define branched covers in all dimensions (see [Rol76, Chapter 10, §B] or [Gol, §3.2]). Describe further examples. Sketch a proof that every closed oriented 3–manifold is a branched cover of S^3, branched over some link ([Ale23, Fei86] or [Rol76, Chapter 10, §G]).

(9) **Branched covers and bridge trisections (smooth case)**
Show how knots in bridge position in 3–manifolds naturally give Heegaard splittings of their branched covers, and then show how this generalizes to knotted surfaces in bridge trisected position giving rise to trisections of their branched covers. See [MZ17, §2.6] and [LCM20, §3].

(10) **Trisections of algebraic surfaces as application of bridge trisections of algebraic curves**
Explain that algebraic surfaces can be seen as branched covers of simpler algebraic surfaces branched over complex curves, and show how this allows us to produce trisection diagrams of many interesting 4–manifolds. The goal would be to explain as many of the trisection diagrams in [LCM20, §10] as is reasonable, working backwards through the paper to find out exactly how much of §5 through §9 is needed to explain the diagrams we are interested in.

(11) **Putting symplectic surfaces in $\mathbb{CP}^2$ into bridge position - part 1**
The goal of this lecture and the next is to prove Theorem 1.2 of [LCMS21], namely that every symplectic surface in $\mathbb{CP}^2$ can be put into bridge trisected position with respect to the standard genus 1 trisection of $\mathbb{CP}^2$. The bulk of this proof is in §5, with some preliminary material in §3.1. The students assigned to these two lectures should work together to decide how to divide up the material of §5. In other words, this lecture is §3.1 and part of §5 and ...

(12) **Symplectic surfaces, $\mathbb{CP}^2$, bridge position - part 2**
... this lecture is the rest of §5.

(13) **Contact structures in dimension 3, Liouville vector fields, and symplectizations**
Define and give examples of contact structures in dimension 3, show how Liouville vector fields transverse to hypersurfaces in symplectic manifolds induce contact structures on the hypersurfaces, define the symplectization of a contact structure, and show that the neighborhood of a contact hypersurface is symplectomorphic to its symplectization. In general dimensions this is explained in [MS17, § 3.5], and the presenters can combine this with Etnyre's notes [Etn] on contact topology for the 3–dimensional perspective, bearing in mind that these latter notes do not discuss much about the relationship with symplectic geometry.

(14) **Weinstein 4–manifolds, Liouville manifolds, handle decompositions**
Weinstein 4–manifolds are symplectic 4–manifolds built handle-by-handle starting with the standard symplectic 4–ball and attaching "symplectic handles". Explain what the preceding sentence means. Weinstein's original paper [Wei91] is concise and readable, and is a good place to start before looking at [CE12, Chapter 11]. Compare Weinstein structures to the more general Liouville structures and explain that there could be a nontrivial difference.

(15) **Existence of Weinstein trisections - part 1**
The goal of these last two lectures is to prove Theorem 1.1 of [LCMS21], that every symplectic 4–manifold admits a Weinstein trisection. This first lecture should state the definition of Weinstein trisection, remind us that (or explain that) the standard trisection of $\mathbb{CP}^2$ is a Weinstein trisection, and then cover [LCMS21, § 4]

(16) **Existence of Weinstein trisections - part 2**
Present [LCMS21, § 6] and as much of §7 of that paper as possible.

References

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